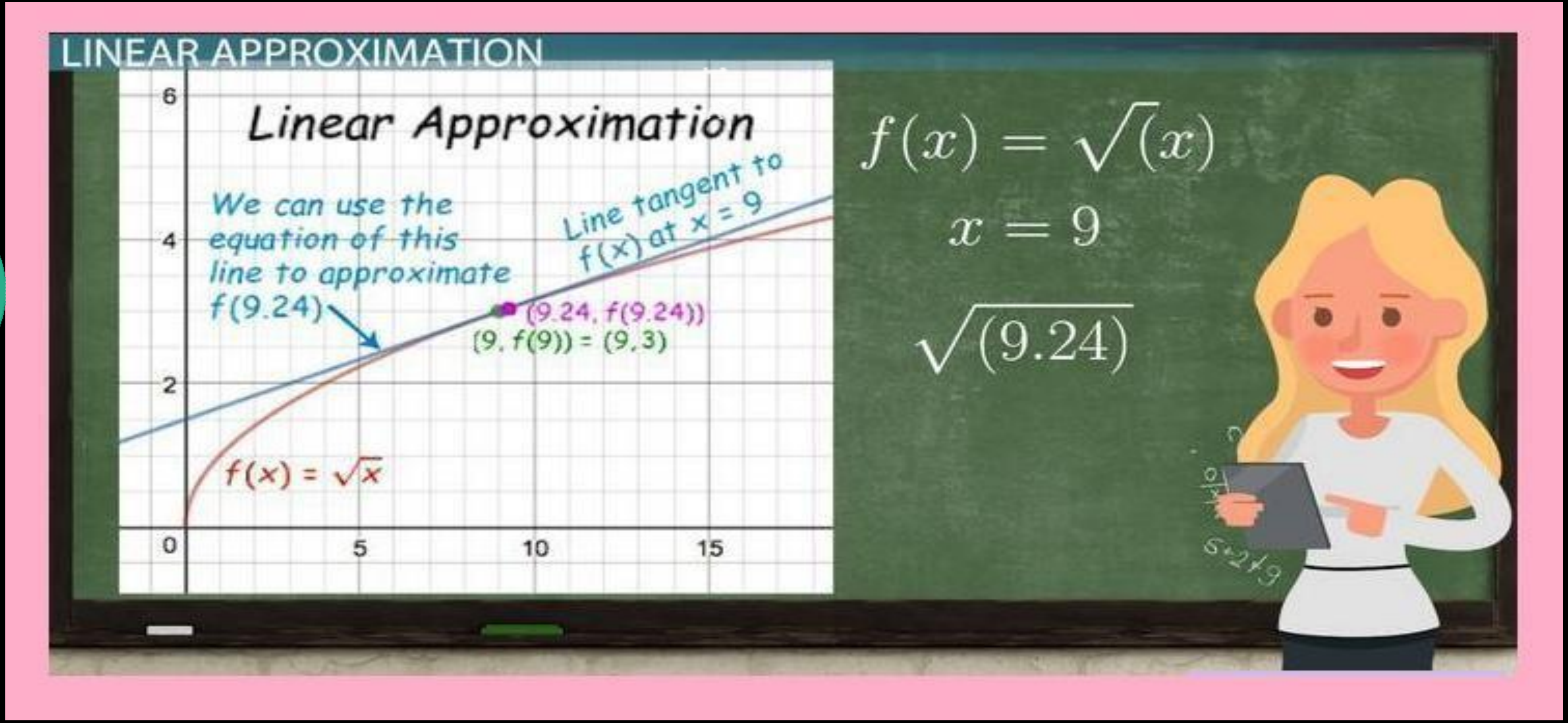
APPLICATION OF DERIVATIVES - MODULE 4

APPROXIMATIONS

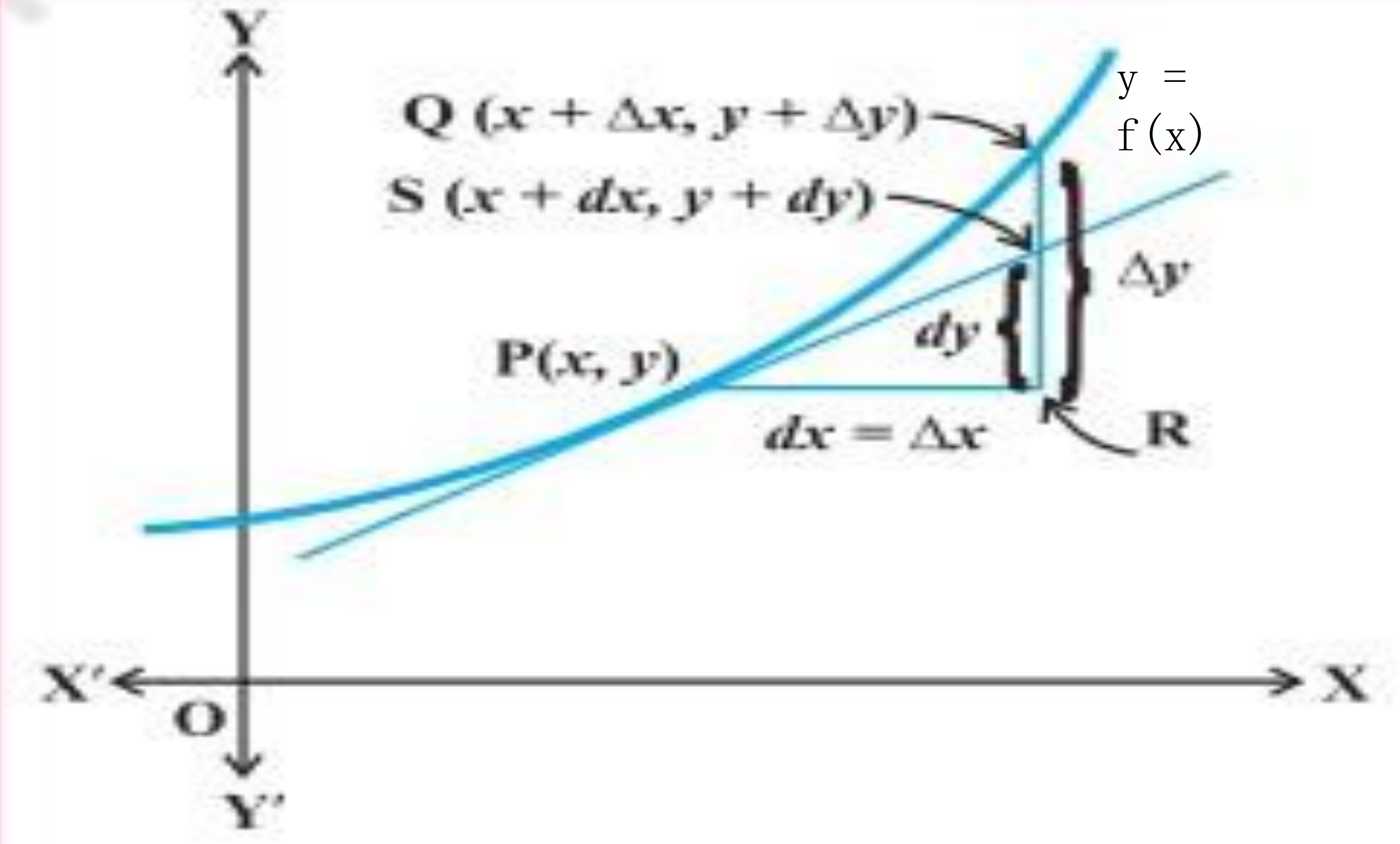
$\Delta x = dx$

$\boldsymbol{\Delta y \approx dy}$



WHAT'S THE THEORY BEHIND APPROXIMATION

f (x)



Now the increment in y corresponding to the increment in x, denoted by Δy, is given by

WHAT'S THE THEORY BEHIND APPROXIMATION

$\Delta y = f(x + \Delta x) - f(x)$

We define the following:

(i) The differential of x, denoted by dx, is defined by $dx = \Delta x$.

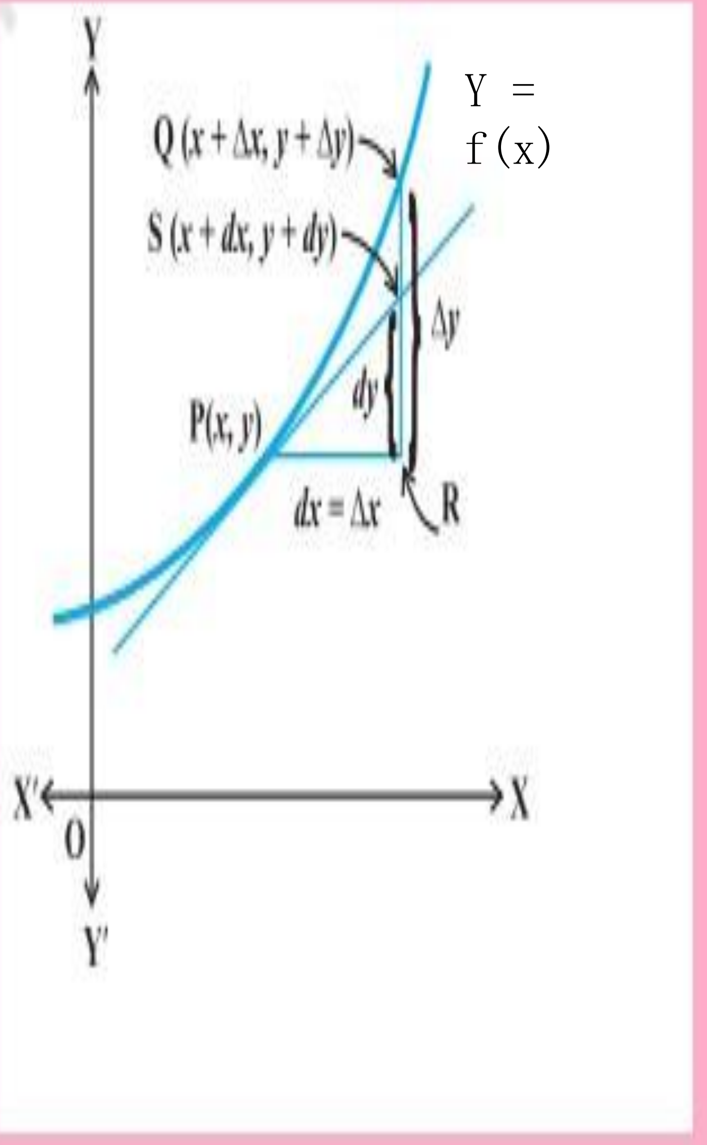
(ii) The differential of y, denoted by dy, is defined by $dy = f'(x)\,dx$ or

$dy = (dy/dx) * \Delta x$

If $dx = \Delta x$ is relatively small when compared to with x then dy is a good approximation of Δy and $dy \approx \Delta y$.

DIFFERENTIAL OF THE DEPENDENT VARIABLE IS NOT EQUAL TO THE INCREMENT OF THE VARIABLE,WHERE AS THE DIFFERENTIAL OF THE INDEPENDENT VARIABLE IS EQUAL TO THE INCREMENT OF THE VARIABLES

$\Delta y \approx dy$
$\Delta x = dx$



FIND THE APPROXIMATE VALUE OF $\sqrt{25.3}$

(i) $\sqrt{25.3}$

Consider $y=\sqrt{x}$. Let $x=25$ and $\Delta x=0.3$.

Then,

$$\Delta y=\sqrt{x+\Delta x}-\sqrt{x}=\sqrt{25.3}-\sqrt{25}=\sqrt{25.3}-5$$

$$\Rightarrow \sqrt{25.3}=\Delta y+5$$

Now, dy is approximately equal to Δy, as $dx = \Delta x$

$$\sqrt{25.3}\approx dy+5$$

$$\approx\left(\frac{dy}{dx}\right)\Delta x+5=\frac{1}{2\sqrt{x}}(0.3)+5=\frac{1}{2\sqrt{25}}(0.3)+5=0.03+5$$

$$\approx 5.03$$

Hence, the approximate value of $\sqrt{25.3}$ is $0.03+5=5.03$.

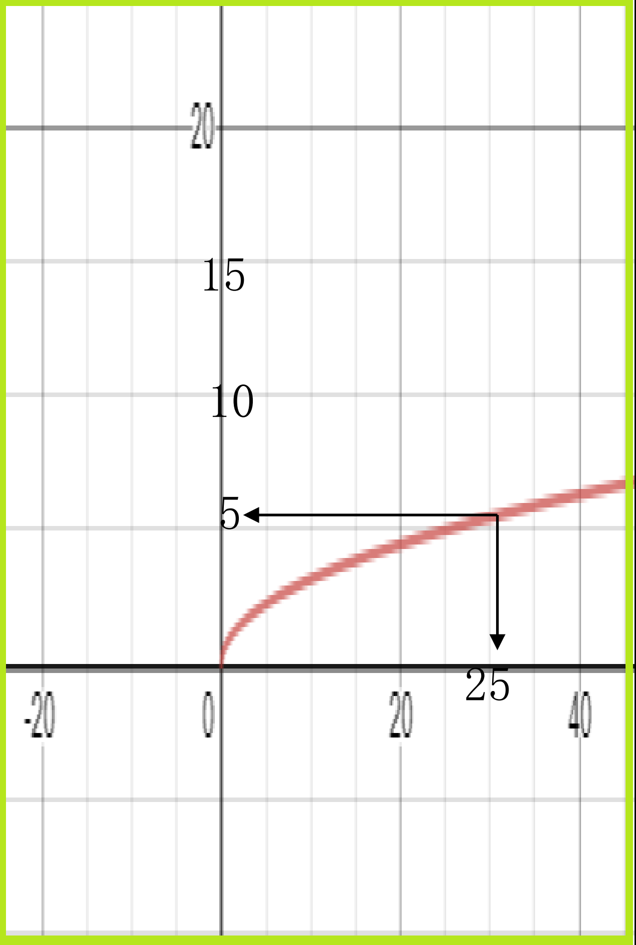
$$y = f(x) = \sqrt{x}$$

$$y+\Delta y = f(x+\Delta x)$$

$$y+\Delta y = \sqrt{x+\Delta x}$$

$$\Delta y = \sqrt{x+\Delta x} - y$$

$$\Delta y = \sqrt{x+\Delta x} - \sqrt{x}$$



FIND THE APPROXIMATE VALUE OF $(0.009)^{\frac{1}{3}}$

$\Delta y = f(x + \Delta x) - f(x)$

y = $(0.009)^{\frac{1}{3}}$

Consider $y = x^{\frac{1}{3}}$. Let $x = 0.008$ and $\Delta x = 0.001$.
Then,

$$\Delta y = (x + \Delta x)^{\frac{1}{3}} - (x)^{\frac{1}{3}} = (0.009)^{\frac{1}{3}} - (0.008)^{\frac{1}{3}} = (0.009)^{\frac{1}{3}} - 0.2$$

$$\Rightarrow (0.009)^{\frac{1}{3}} = 0.2 + \Delta y$$

Now, dy is approximately equal to Δy and is given by,

$$\Rightarrow (0.009)^{\frac{1}{3}} \approx dy + 0.2$$

$$\approx \left(\frac{dy}{dx}\right)\Delta x + 0.2$$

$$\approx \frac{1}{3(x)^{\frac{2}{3}}}(\Delta x) + 0.2$$

as $y = x^{\frac{1}{3}}$

$$\frac{dy}{dx} = \frac{1}{3(x)^{\frac{2}{3}}}$$

$$\approx \frac{1}{3 \times 0.04}(0.001) + 0.2 = \frac{0.001}{0.12} + 0.2 = 0.008 + 0.2$$

Hence, the approximate value of $(0.009)^{\frac{1}{3}}$ is $0.2 + 0.008 = 0.208$.

$y = (0.999)^{\frac{1}{10}}$

Δx *is* $-ve$

Consider $y = (x)^{\frac{1}{10}}$. Let $x = 1$ and $\Delta x = -0.001$.

Then,

$\Delta y = (x + \Delta x)^{\frac{1}{10}} - (x)^{\frac{1}{10}} = (0.999)^{\frac{1}{10}} - 1$

$\Delta y = f(x + \Delta x) - f(x$

$\Rightarrow (0.999)^{\frac{1}{10}} = 1 + \Delta y$

Now, dy is approximately equal to Δy and is given by,

$\Rightarrow (0.999)^{\frac{1}{10}}$ dy + 1

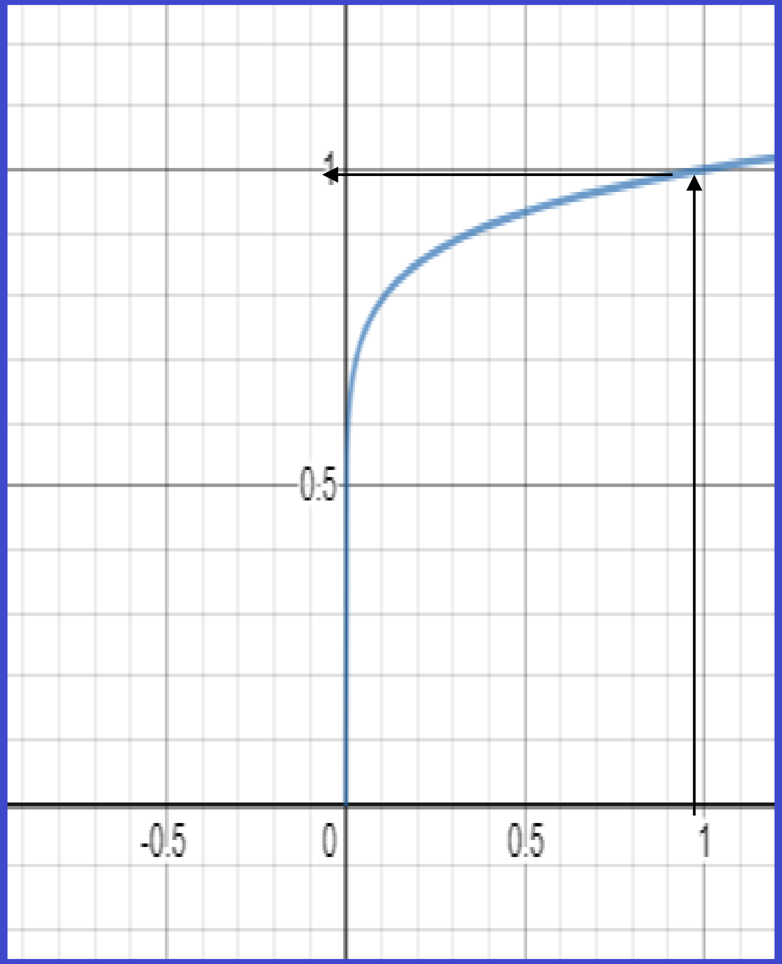
$\left(\frac{dy}{dx}\right)\Delta x + 1 \quad = \frac{1}{10(x)^{\frac{9}{10}}}(\Delta x) + 1$

as $y = (x)^{\frac{1}{10}}$

$\frac{dy}{dx} = \frac{1}{10(x)^{\frac{9}{10}}}$

$\frac{1}{10}(-0.001) + 1 \quad = -0.0001 + 1 = 0.9999$

Hence, the approximate value of $(0.999)^{\frac{1}{10}}$ is $1 + (-0.0001) = 0.9999$.



Question

Find the approximate value of f (2.01), where $f(x)=4x^2+5x+2$.

Solution

Let $x=2$ and $\Delta x=0.01$. Then, we have:

$f(2.01)=f(x+\Delta x)=4(x+\Delta x)^2+5(x+\Delta x)+2$

Now, $\Delta y=f(x+\Delta x)-f(x)$

$f(x+\Delta x)=f(x)+\Delta y$

$\approx f(x)+f'(x).\Delta x$ $\qquad (as\ dx=\Delta x)$

$\Rightarrow f(2.01)\approx(4x^2+5x+2)+(8x+5)\Delta x$

$\approx [4(2)^2+5(2)+2]+[8(2)+5](0.01)$ $\qquad [as\ x=2,\ \Delta x=0.01]$

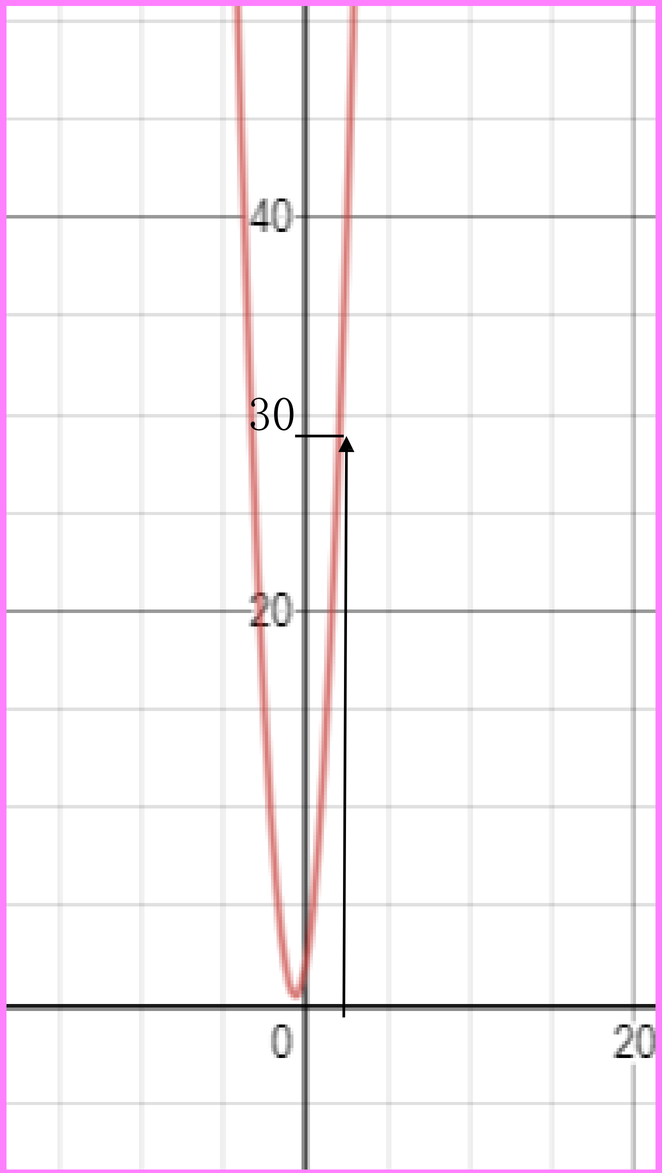
$\approx (16+10+2)+(16+5)(0.01)$

$\approx 28+(21)(0.01)$

$\approx 28+0.21$

≈ 28.21

Hence, the approximate value of $f(2.01)$ is 28.21.



Question

Find the approximate value of $f(5.001)$, where $f(x)=x^3-7x^2+15$.

Solution

Let $x=5$ and $\Delta x=0.001$. Then, we have:

$f(5.001)=f(x+\Delta x)=(x+\Delta x)^3-7(x+\Delta x)^2+15$

Now, $\Delta y=f(x+\Delta x)-f(x)$

$\therefore f(x+\Delta x)=f(x)+\Delta y$

$\approx f(x)+f'(x).\Delta x$ $\quad (as\ dx=\Delta x)$

$\Rightarrow f(5.001)\approx(x^3-7x^2+15)+(3x^2-14x)\Delta x$

$\approx\left[(5)^3-7(5)^2+15\right]+\left[3(5)^2-14(5)\right](0.001)$ $\quad [x=5,\Delta x=0.001]$

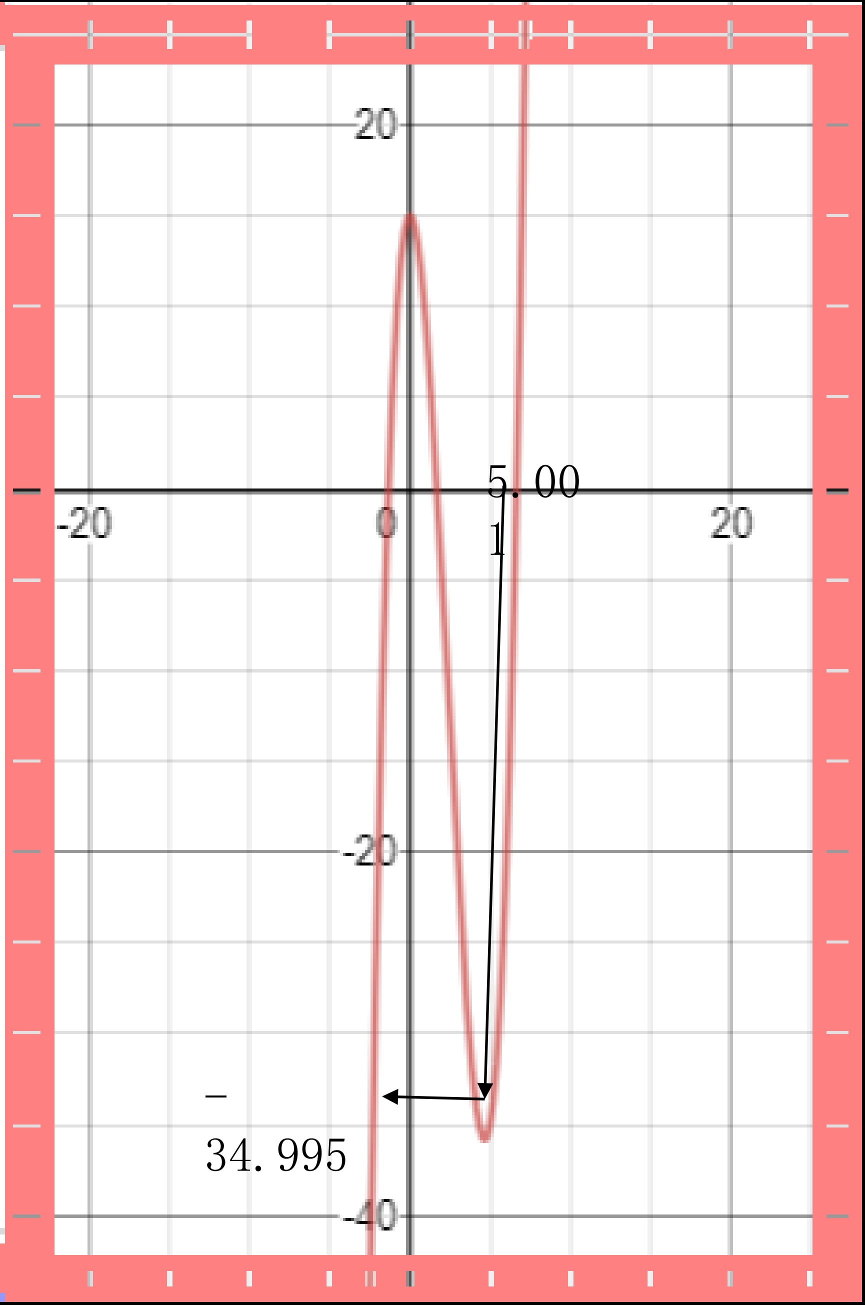
$\approx(125-175+15)+(75-70)(0.001)$

$\approx -35+(5)(0.001)$

$\approx -35+0.005$

≈ -34.995

Hence, the approximate value of $f(5.001)$ is -34.995.



Find the approximate change in the surface area of a cube of side x meters caused by decreasing the side by 1%.

Solution 5:

The surface area of a cube (S) of side x is given by $S = 6x^2$.

When side decreases , surface area decreases

$$\therefore \frac{ds}{dx} = \left(\frac{ds}{dx}\right)\Delta x$$

$\Delta S \approx dS$

$$= (12x)\Delta x$$

$$= (12x)(0.01x) \qquad [as\ 1\%\ of\ x\ is\ 0.01x]$$

$$= 0.12x^2$$

Hence, the approximate change in the surface area of the cube is $0.12x^2\ \text{m}^2$ decrease

Question

If the radius of a sphere is measured as 7 m with an error of 0.02 m, then find the approximate error in calculating its volume.

Solution

Let r be the radius of the sphere and Δr be the error in measuring the radius.

Then,

$r = 7m$ and $\Delta r = 0.02m$

Now, the volume V of the sphere is given by,

$$V = \frac{4}{3}\pi r^3$$

$$\therefore \frac{dV}{dr} = 4\pi r^2$$

$$\therefore dV = \left(\frac{dV}{dr}\right)\Delta r$$

$$= \left(4\pi r^2\right)\Delta r$$

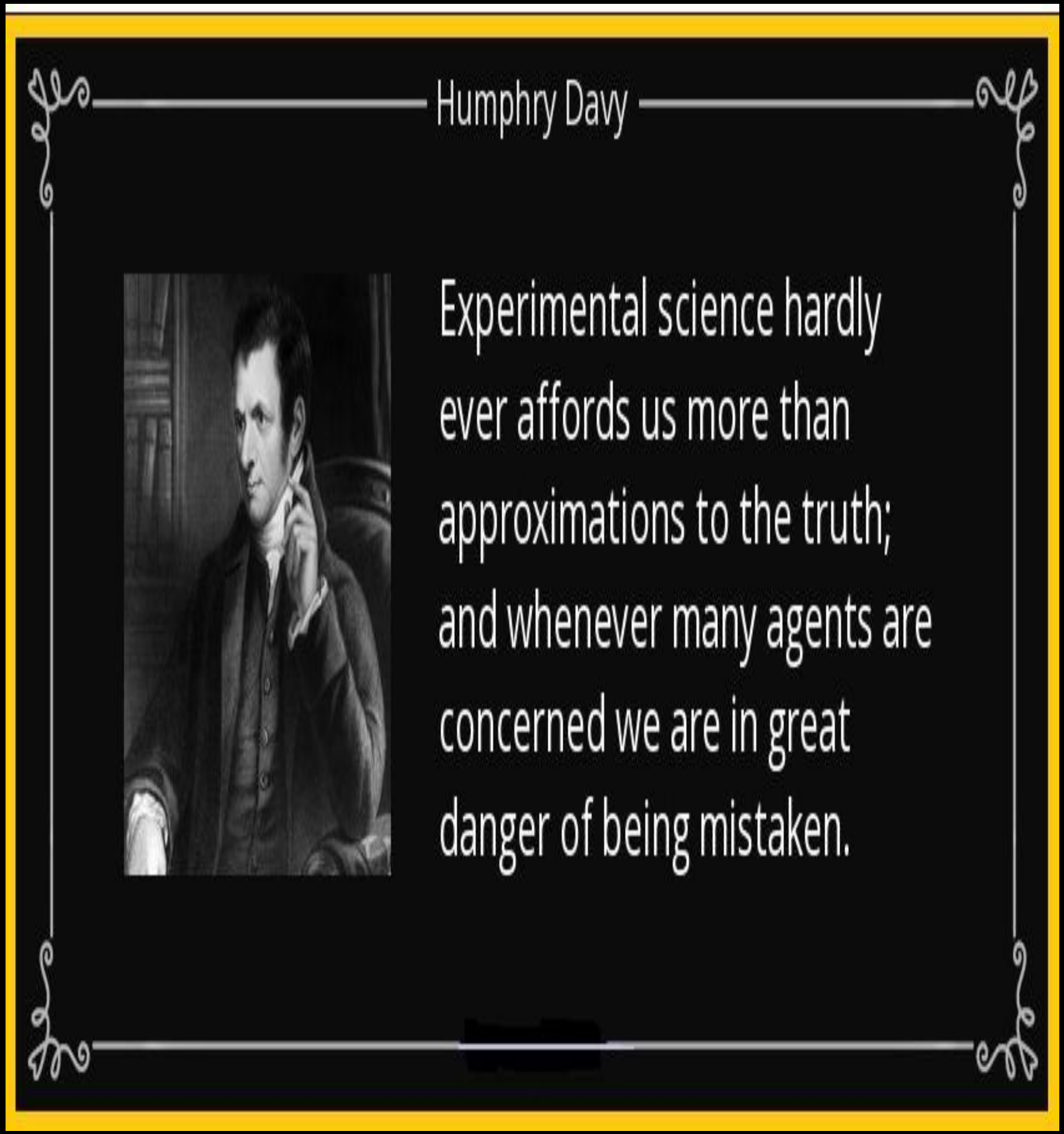
$$= 4\pi (7)^2 (0.02) m^3 = 3.92\pi m^3$$

Hence, the approximate error in calculating the volume is 3.92π m^3.

$$\Delta V \approx dV$$

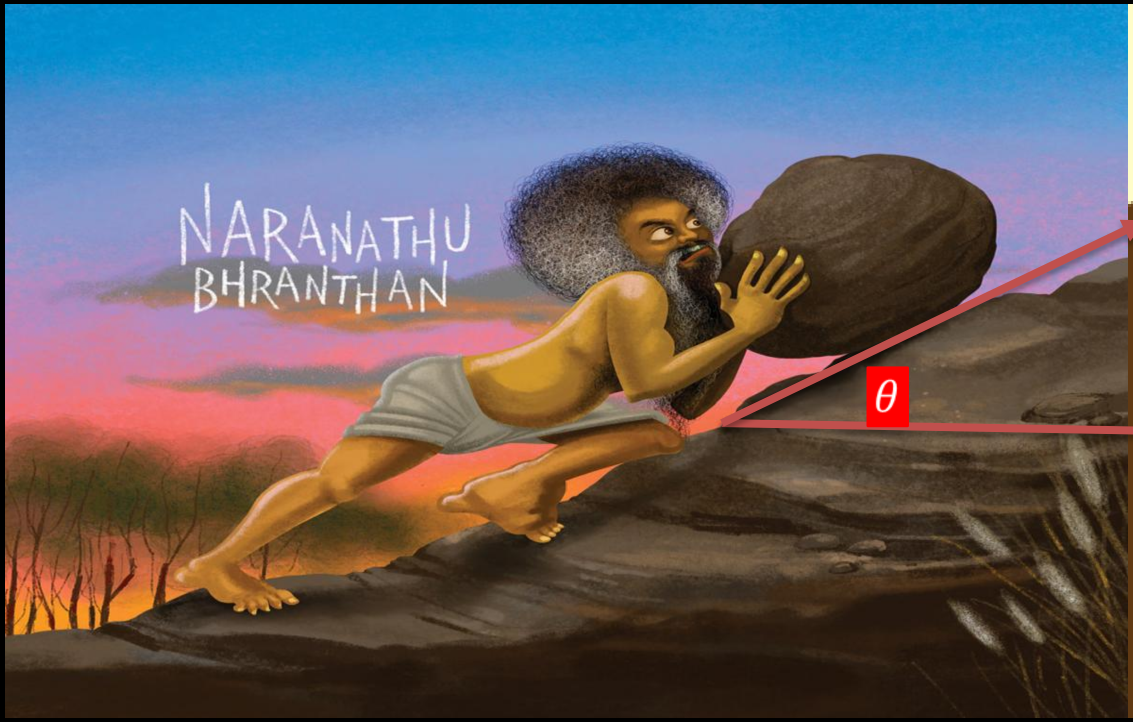
HOME WORK

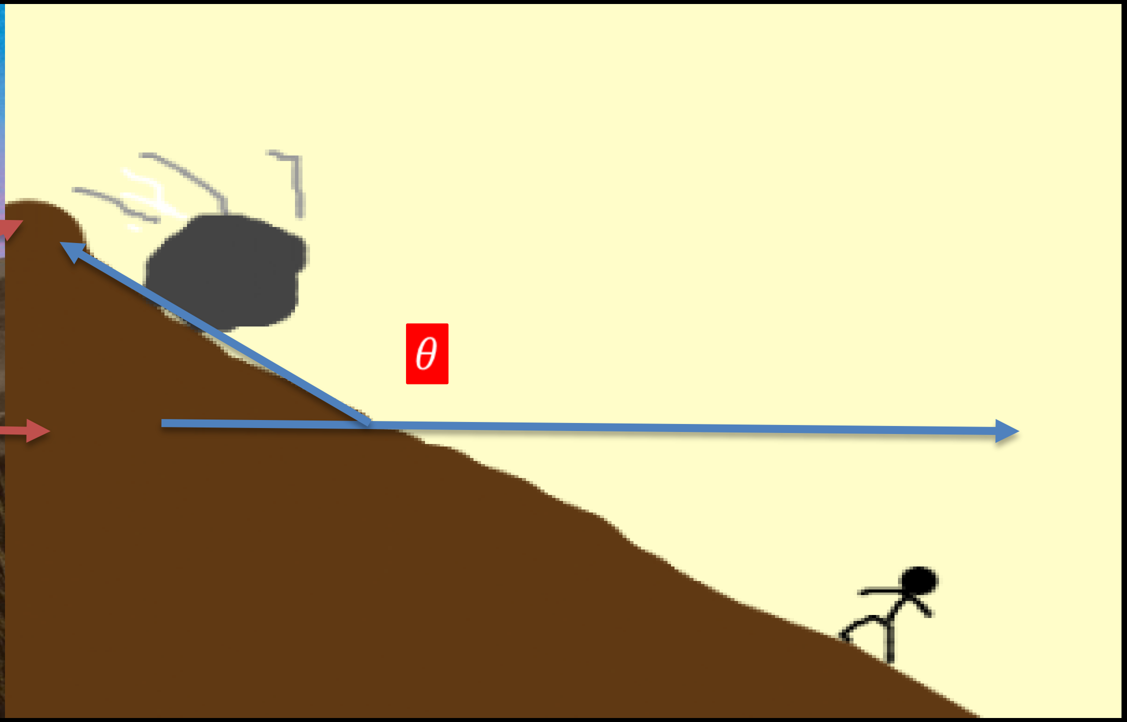
- EX
  6.4
- Q1 : (iii), (xiv) , (xv)
- Q7, Q8, Q9

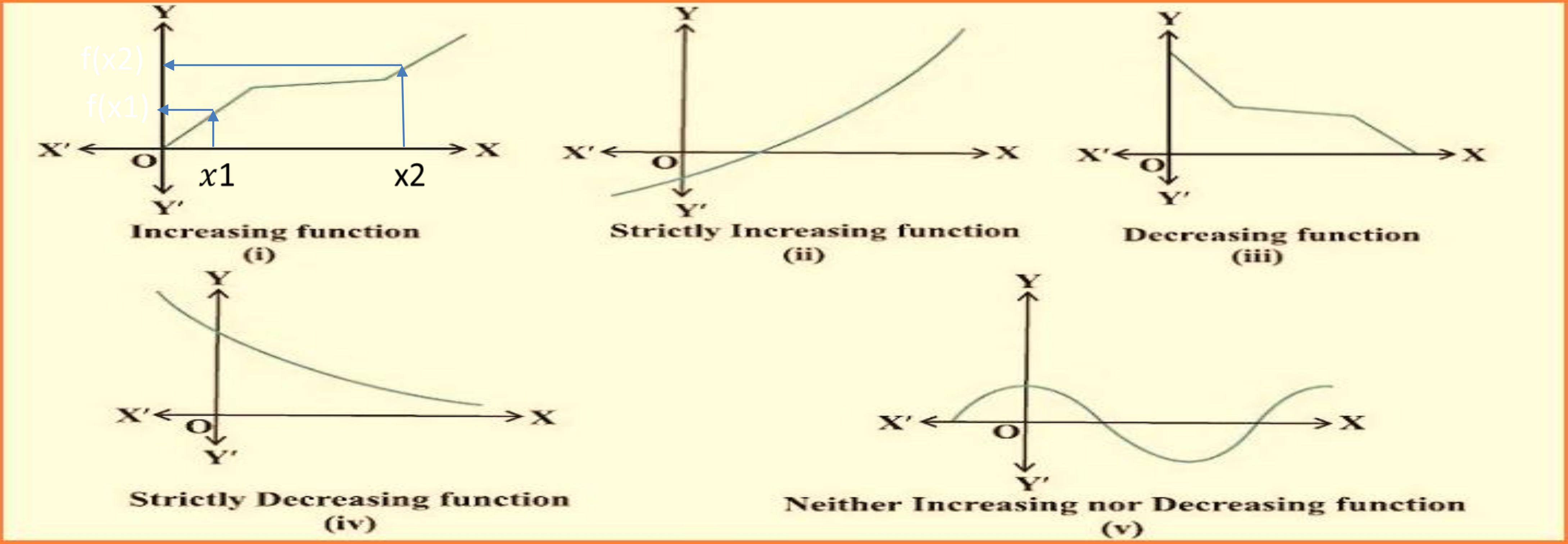


APPLICATION OF DERIVATIVES-MODULE 5

INCREASING AND DECREASING FUNCTIONS







Definition 1 Let I be an open interval contained in the domain of a real valued function f. Then f is said to be

(i) increasing on I if $x_1 < x_2$ in I $\Rightarrow f(x_1) \leq f(x_2)$ for all $x_1, x_2 \in$ I.

(ii) strictly increasing on I if $x_1 < x_2$ in I $\Rightarrow f(x_1) < f(x_2)$ for all $x_1, x_2 \in$ I.

(iii) decreasing on I if $x_1 < x_2$ in I $\Rightarrow f(x_1) \geq f(x_2)$ for all $x_1, x_2 \in$ I.

(iv) strictly decreasing on I if $x_1 < x_2$ in I $\Rightarrow f(x_1) > f(x_2)$ for all $x_1, x_2 \in$ I.

Show that the function given by $f(x) = 3x + 17$ is strictly increasing on **R**.

Let x_1 and x_2 be real numbers

Such that

$$x_1 < x_2$$

Multiplying both sides by 3

$$3x_1 < 3x_2$$

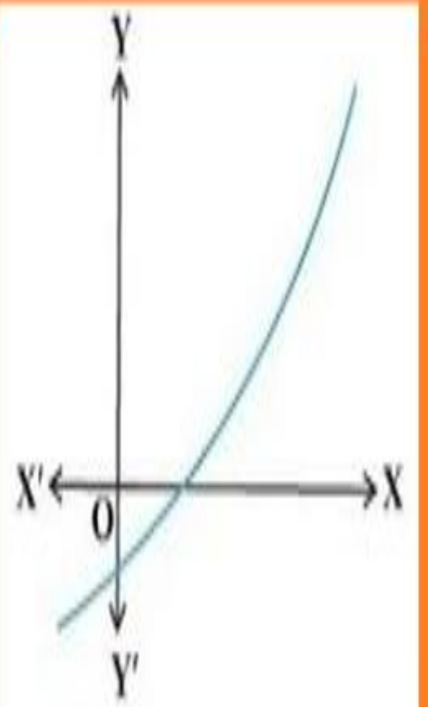
Adding both sides by 17

$$3x_1 + 17 < 3x_2 + 17$$

$$f(x_1) < f(x_2)$$

Hence when $x_1 < x_2$, $f(x_1) < f(x_2)$

Thus, **f(x) is strictly increasing on R.**



Strictly Increasing function

(ii)

WORKING RULE

Theorem 1 Let f be continuous on $[a, b]$ and differentiable on the open interval (a,b). Then

(a) f is increasing in $[a,b]$ if $f'(x) > 0$ for each $x \in (a, b)$

(b) f is decreasing in $[a,b]$ if $f'(x) < 0$ for each $x \in (a, b)$

(c) f is a constant function in $[a,b]$ if $f'(x) = 0$ for each $x \in (a, b)$

Remarks

(i) f is strictly increasing in (a, b) if $f'(x) > 0$ for each $x \in (a, b)$

(ii) f is strictly decreasing in (a, b) if $f'(x) < 0$ for each $x \in (a, b)$

(iii) A function will be increasing (decreasing) in **R** if it is so in every interval of **R**.

Example Show that the function f given by

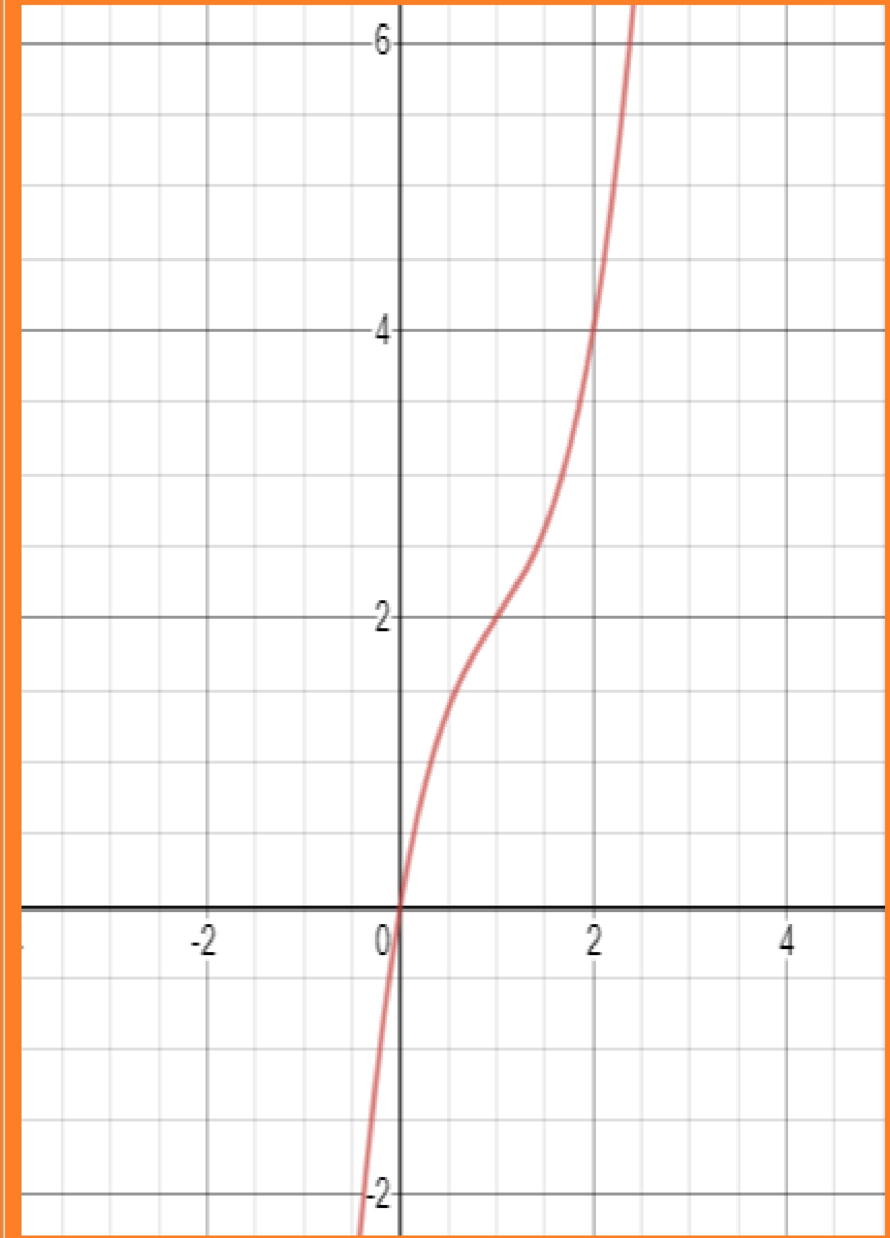
$$f(x) = x^3 - 3x^2 + 4x,\ x \in \mathbf{R}$$

is strictly increasing on $\mathbf{R}$.

Solution Note that

$$\begin{aligned} f'(x) &= 3x^2 - 6x + 4 \\ &= 3(x^2 - 2x + 1) + 1 \\ &= 3(x-1)^2 + 1 > 0, \text{ in every interval of } \mathbf{R} \end{aligned}$$

Therefore, the function f is strictly increasing on $\mathbf{R}$.



Example Prove that the function given by $f(x) = \cos x$ is

(a) strictly decreasing in $(0, \pi)$

(b) strictly increasing in $(\pi, 2\pi)$, and

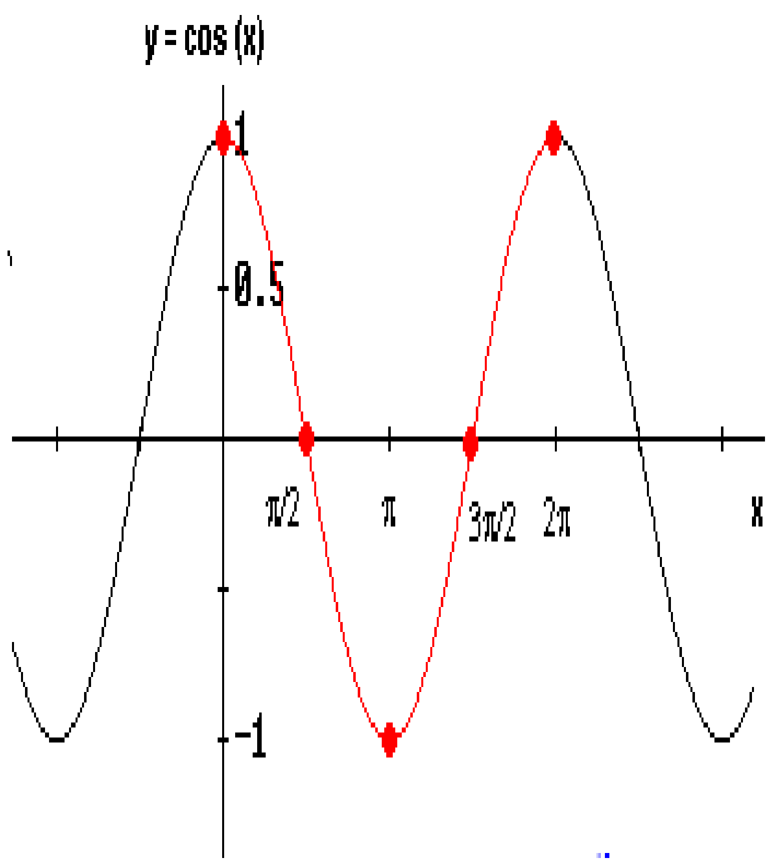
(c) neither increasing nor decreasing in $(0, 2\pi)$.

Solution Note that $f'(x) = -\sin x$

(a) Since for each $x \in (0, \pi)$, $\sin x > 0$, we have $f'(x) < 0$ and so f is strictly decreasing in $(0, \pi)$.

(b) Since for each $x \in (\pi, 2\pi)$, $\sin x < 0$, we have $f'(x) > 0$ and so f is strictly increasing in $(\pi, 2\pi)$.

(c) Clearly by (a) and (b) above, f is neither increasing nor decreasing in $(0, 2\pi)$.



Note One may note that the function in question is neither strictly increasing in $[\pi, 2\pi]$ nor strictly decreasing in $[0, \pi]$. However, since the function is continuous at the end points 0 and π, by Theorem 1, f is increasing in $[\pi, 2\pi]$ and decreasing in $[0, \pi]$.

Example Find the intervals in which the function f given by $f(x) = x^2 - 4x + 6$ is

(a) strictly increasing (b) strictly decreasing

Solution We have

$$f(x) = x^2 - 4x + 6$$

or

$$f'(x) = 2x - 4$$

Therefore, $f'(x) = 0$ gives $x = 2$. Now the point $x = 2$ divides the real line into two disjoint intervals namely, $(-\infty, 2)$ and $(2, \infty)$ (Fig 6.3). In the interval $(-\infty, 2)$, $f'(x) = 2x - 4 < 0$.

Therefore, f is strictly decreasing in this interval. Also, in the interval $(2, \infty)$, $f'(x) > 0$ and so the function f is strictly increasing in this interval.

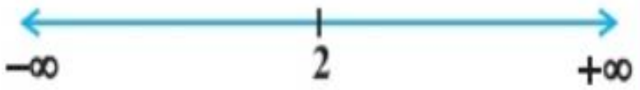
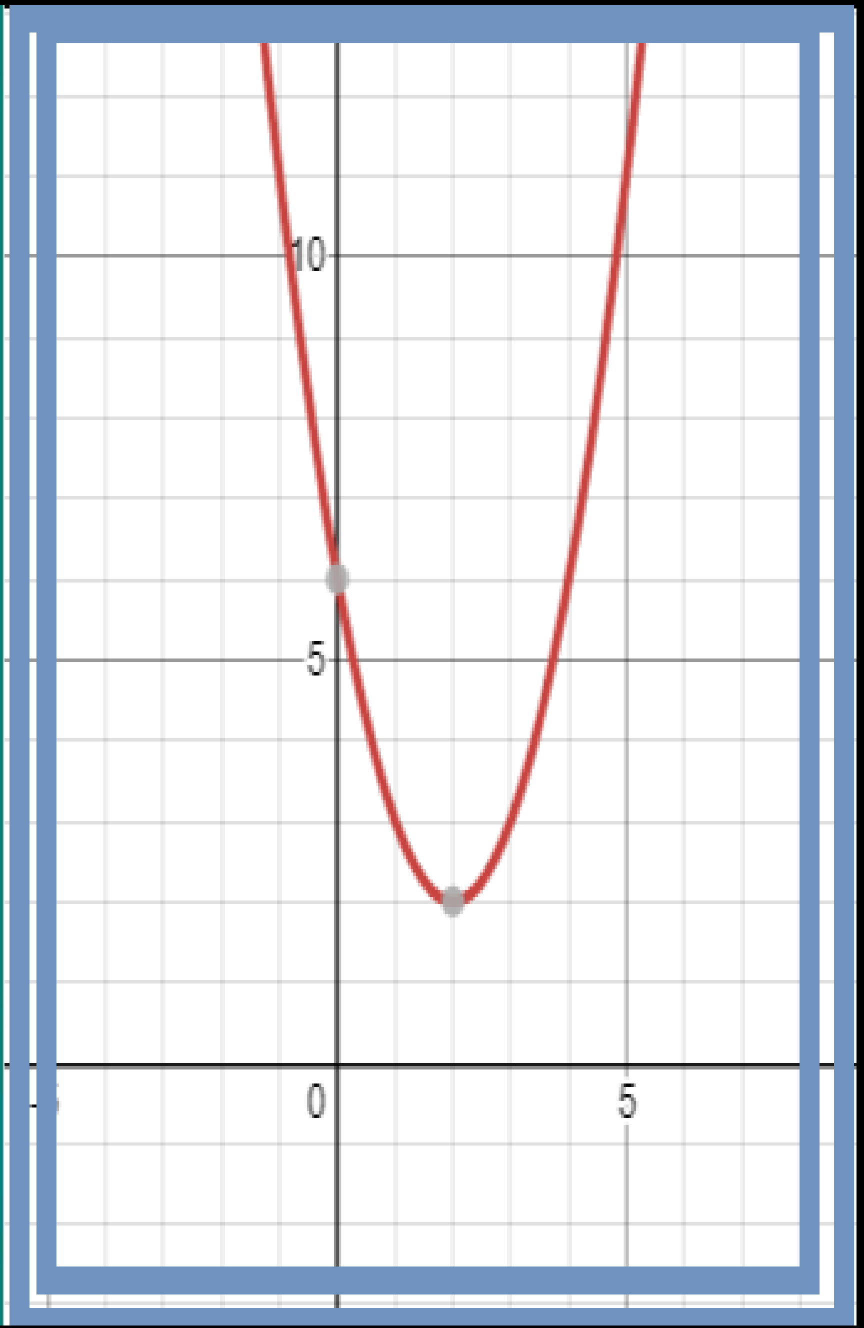


Fig 6.3

> **Note** Note that the given function is continuous at 2 which is the point joining the two intervals. So, by Theorem 1, we conclude that the given function is decreasing in $(-\infty, 2]$ and increasing in $[2, \infty)$.



Example Find the intervals in which the function f given by $f(x) = 4x^3 - 6x^2 - 72x + 30$ is (a) strictly increasing (b) strictly decreasing.

Solution We have

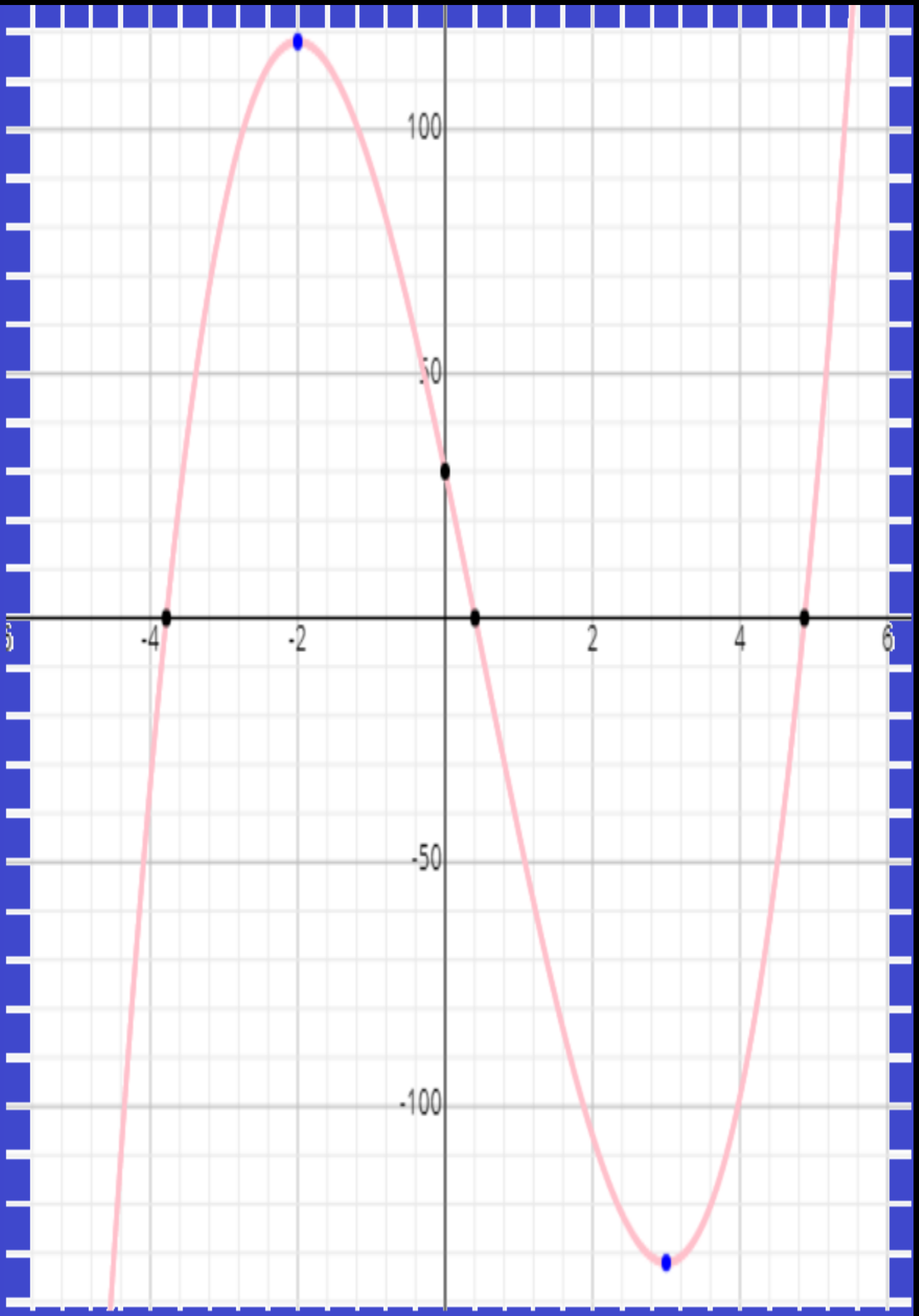
$$f(x) = 4x^3 - 6x^2 - 72x + 30$$

$$f'(x) = 12x^2 - 12x - 72$$
$$= 12(x^2 - x - 6)$$
$$= 12(x - 3)(x + 2)$$

Therefore, $f'(x) = 0$ gives $x = -2, 3$. The points $x = -2$ and $x = 3$ divides the real line into three disjoint intervals, namely, $(-\infty, -2)$, $(-2, 3)$ and $(3, \infty)$.

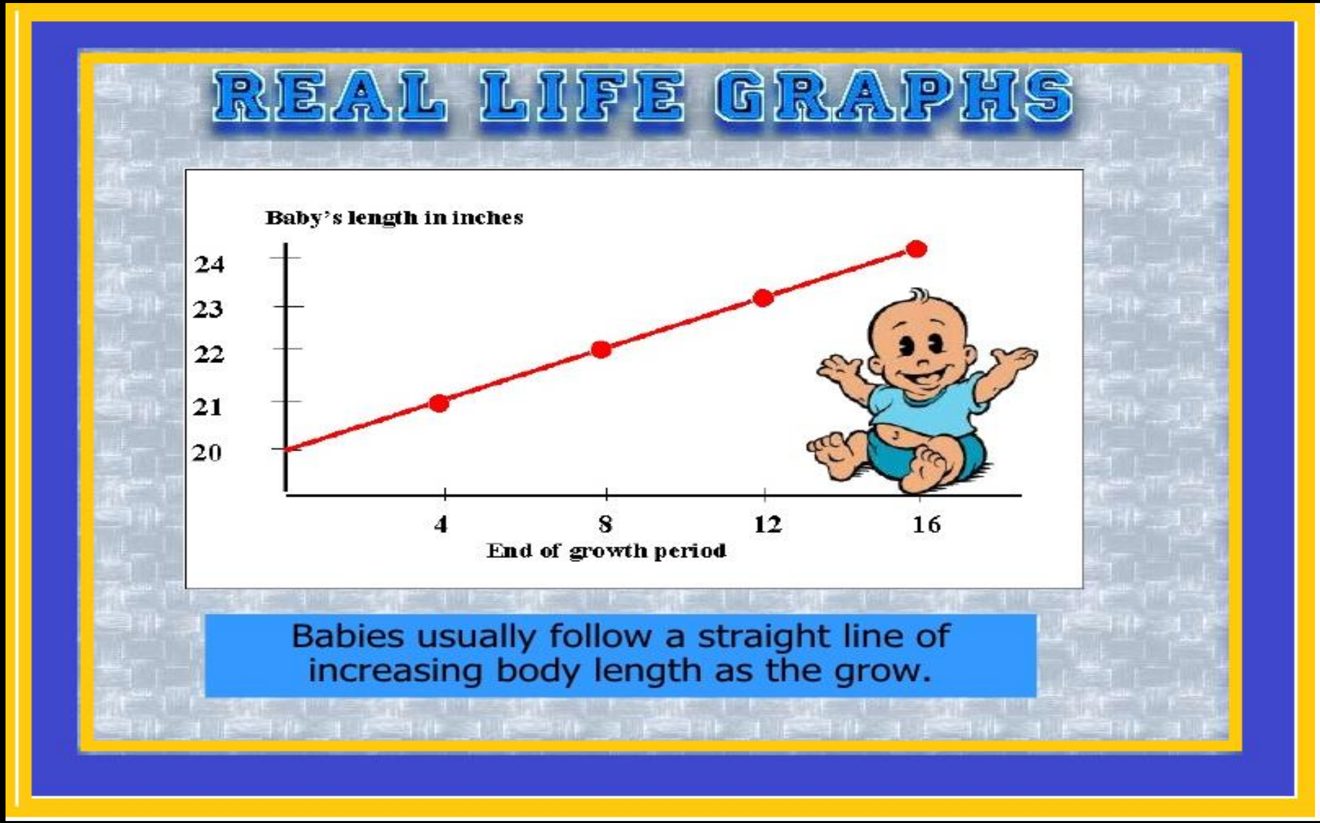
In the intervals $(-\infty, -2)$ and $(3, \infty)$, $f'(x)$ is positive while in the interval $(-2, 3)$, $f'(x)$ is negative. Consequently, the function f is strictly increasing in the intervals $(-\infty, -2)$ and $(3, \infty)$ while the function is strictly decreasing in the interval $(-2, 3)$. However, f is neither increasing nor decreasing in **R**.

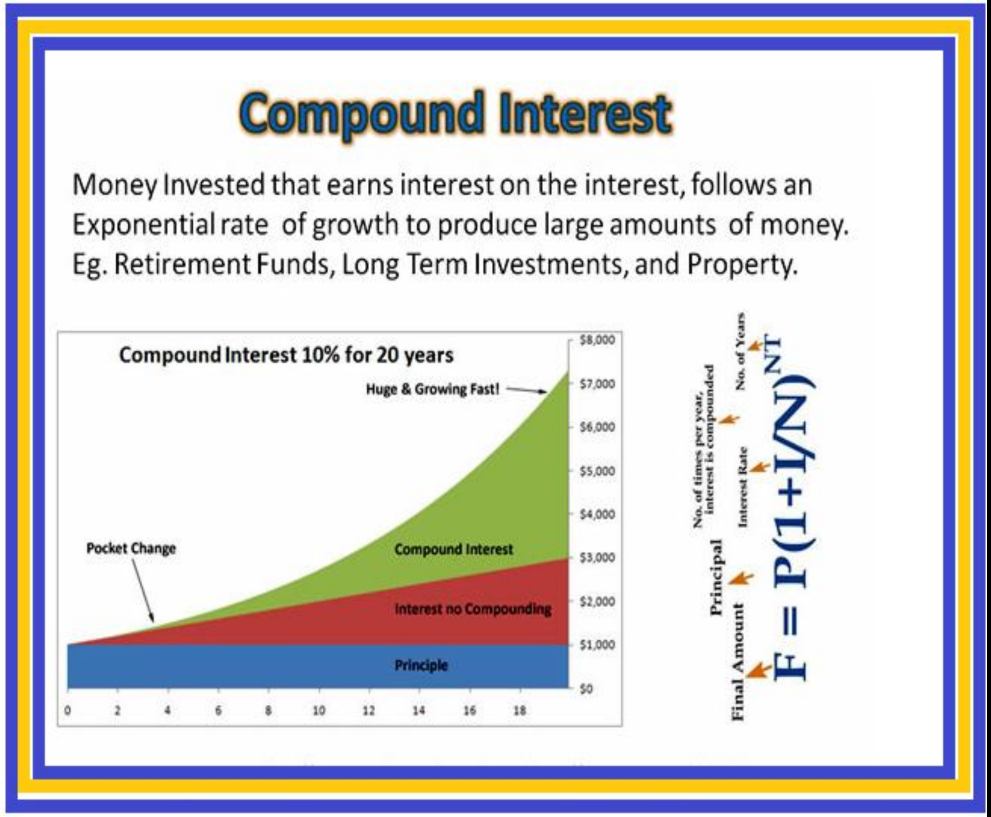
Interval	Sign of $f'(x)$	Nature of function f
$(-\infty, -2)$	$(-)(-) > 0$	f is strictly increasing
$(-2, 3)$	$(-)(+) < 0$	f is strictly decreasing
$(3, \infty)$	$(+)(+) > 0$	f is strictly increasing



APPLICATION OF DERIVATIVES-MODULE 6

INCREASING AND DECREASING FUNCTIONS





Find the intervals in which the following functions are strictly increasing or strictly decreasing:

$-2x^3 - 9x^2 - 12x + 1$

Let $f(x) = -2x^3 - 9x^2 - 12x + 1$

$\therefore \quad f'(x) = -6x^2 - 18x - 12 = -6(x^2 + 3x + 2) = -6(x + 1)(x + 2)$

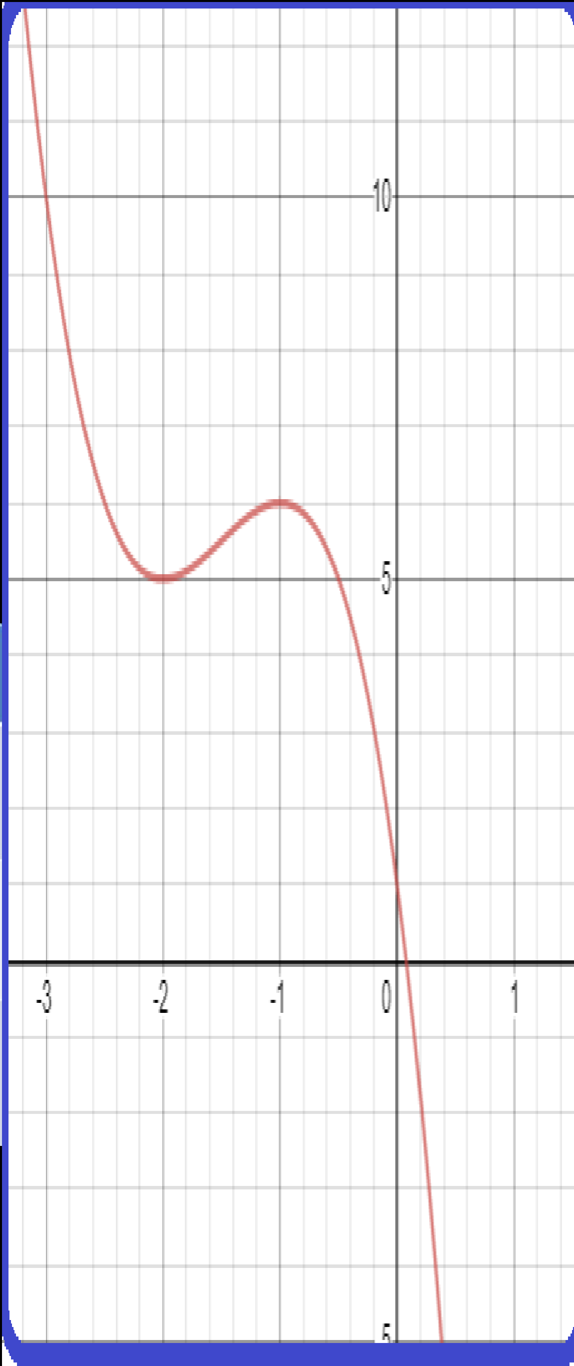
$f'(x) = 0$ gives us $-6(x + 1)(x + 2) = 0 \Rightarrow x = -1, -2$

The points $x = -2, -1$ divide the real line into three intervals $(-\infty, -2)$, $(-2, -1)$, $(-1, \infty)$

Intervals	Sign of $f'(x) = -6(x+1)(x+2)$	Nature of function f
$x \in (-\infty, -2)$	$(-)(-)(-) = (+)(-) = (-) < 0$	f is strictly decreasing
$x \in (-2, -1)$	$(-)(-)(+) = (+)(+) = (+) > 0$	f is strictly increasing
$x \in (-1, \infty)$	$(-)(+)(+) = (-)(+) = (-) < 0$	f is strictly decreasing

Hence, f is **strictly increasing for** $(-2, -1)$.

& **strictly decreasing for** $(-\infty, -2) \cup (-1, \infty)$



Question 7:

Show that $y = \log(1+x) - \frac{2x}{2+x}, x > -1$, is an increasing function of x throughout its domain.

Answer 7:

$$y = \log(1+x) - \frac{2x}{2+x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1+x} - \frac{(2+x).2 - 2x(1)}{(2+x)^2} = \frac{1}{1+x} - \frac{4}{(2+x)^2} = \frac{(2+x)^2 - 4(1+x)}{(1+x)(2+x)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{4 + x^2 + 4x - 4 - 4x}{(1+x)(2+x)^2} = \frac{x^2}{(1+x)(2+x)^2}$$

$x^2 > 0$ and $(2+x)^2 > 0$, as these are perfect square and $(1+x) > 0$ as $x > -1$.

Therefore, $\frac{dy}{dx} > 0$, if $x > -1$. Hence, the function is increasing throughout its domain.

What is the domain of this function ?

Question 9:

Prove that $y=\dfrac{4\sin\theta}{(2+\cos\theta)}-\theta$ is an increasing function of θ in $\left[0,\dfrac{\pi}{2}\right]$.

Solution 9:

We have,

$$y=\frac{4\sin\theta}{(2+\cos\theta)}-\theta$$

$$\therefore \frac{dy}{d\theta}=\frac{(2+\cos\theta)(4\cos\theta)-4\sin\theta(-\sin\theta)}{(2+\cos\theta)^2}-1$$

$$=\frac{8\cos\theta+4\cos^2\theta+4\sin^2\theta}{(2+\cos\theta)^2}-1$$

$$=\frac{8\cos\theta+4}{(2+\cos\theta)^2}-1$$

Now,

$$\frac{dy}{d\theta}=\frac{8\cos\theta+-(4+\cos^2\theta+4\cos\theta)}{(2+\cos\theta)2}=\frac{4\cos\theta-\cos^{2\theta}}{(2+\cos\theta)^2}=\frac{\cos(4-\cos\theta)}{(2+\cos\theta)^2}$$

In interval $\left[0,\dfrac{\pi}{2}\right]$, we have $\cos\theta>0$, Also $4>\cos\theta\Rightarrow 4-\cos\theta>0$.

$\therefore \cos\theta(4-\cos\theta)>0$ and also $(2+\cos\theta)^2>0$

$$\Rightarrow \frac{\cos\theta(4-\cos\theta)}{(2+\cos\theta)^2}>0$$

$$\Rightarrow \frac{dy}{dx}>0$$

Therefore, y is strictly increasing in interval $\left(0,\dfrac{\pi}{2}\right)$

Also, the given function is continuous at $x=0$ and $x=\dfrac{\pi}{2}$.

Hence, y is increasing in interval $\left[0,\dfrac{\pi}{2}\right]$.

Show that $f(x) = \tan^{-1}(\sin x + \cos x)$ is a decreasing function on the interval $(\pi/4, \pi/2)$.

Solution:

Given $f(x) = \tan^{-1}(\sin x + \cos x)$

$$\Rightarrow f'(x) = \frac{d}{dx}(\tan^{-1}(\sin x + \cos x))$$

$$\Rightarrow f'(x) = \frac{1}{1 + (\sin x + \cos x)^2} \times (\cos x - \sin x)$$

$$f'(x) = \frac{(\cos x - \sin x)}{1 + (\sin x + \cos x)^2}$$

Now, as given

$$x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$$

$\Rightarrow \cos x - \sin x < 0$; as here cosine values are smaller than sine values for same angle

$$f'(x) = \frac{(\cos x - \sin x)}{1 + (\sin x + \cos x)^2}$$

$\Rightarrow f'(x) < 0$

Hence, Condition for f(x) to be decreasing

Thus f(x) is decreasing on interval $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

Find the intervals in which f(x) = sin x − cos x, where 0 < x < 2π is increasing or decreasing.

Solution:

Given f (x) = sin x – cos x

$\Rightarrow f'(x) = \frac{d}{dx}(\sin x - \cos x)$

$\Rightarrow f'(x) = \cos x + \sin x$

For f(x) let us find critical point, we must have

$\Rightarrow f'(x) = 0$

$\Rightarrow \cos x + \sin x = 0$

$\Rightarrow \tan(x) = -1$

$\Rightarrow x = \frac{3\pi}{4}, \frac{7\pi}{4}$

INTERVALS	SIGN OF f'(x)	NATURE OF THE FUNCTION
$(0, \frac{3\pi}{4})$	+	INCREASING
$(\frac{3\pi}{4},, \frac{7\pi}{4})$	-	DECREASING
$\left(\frac{7\pi}{4}, 2\pi\right)$	+	INCREASING

$f(x) = \frac{3}{10}x^4 - \frac{4}{5}x^3 - 3x^2 + \frac{36}{5}x + 11$

is (a) strictly increasing (b) strictly decreasing

$$f(x) = \frac{3}{10}x^4 - \frac{4}{5}x^3 - 3x^2 + \frac{36}{5}x + 11$$

Finding $f'(x)$

$$f'(x) = \frac{3}{10} \times 4x^3 - \frac{4}{5} \times 3x^2 - 3 \times 2x + \frac{36}{5} + 0$$

$$f'(x) = \frac{12}{10}x^3 - \frac{12}{5}x^2 - 6x + \frac{36}{5}$$

$$f'(x) = 6\left(\frac{x^3 - 2x^2 - 5x + 6}{5}\right)$$

$$= \frac{6}{5}(x^3 - 2x^2 - 5x + 6)$$

synthetic division :

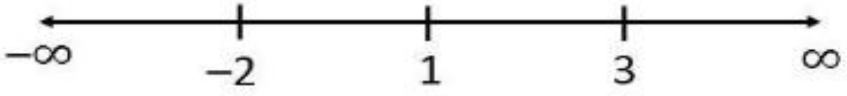
1	1	-2	-5	6
	↓	1	-1	-6
	1	-1	-6	0

$$= \frac{6}{5}(x-1)(x^2 - x - 6)$$

$$= \frac{6}{5}(x-1)(x+2)(x-3)$$

$f'(x) = 0$

Hence x = −2 . 1 & 3



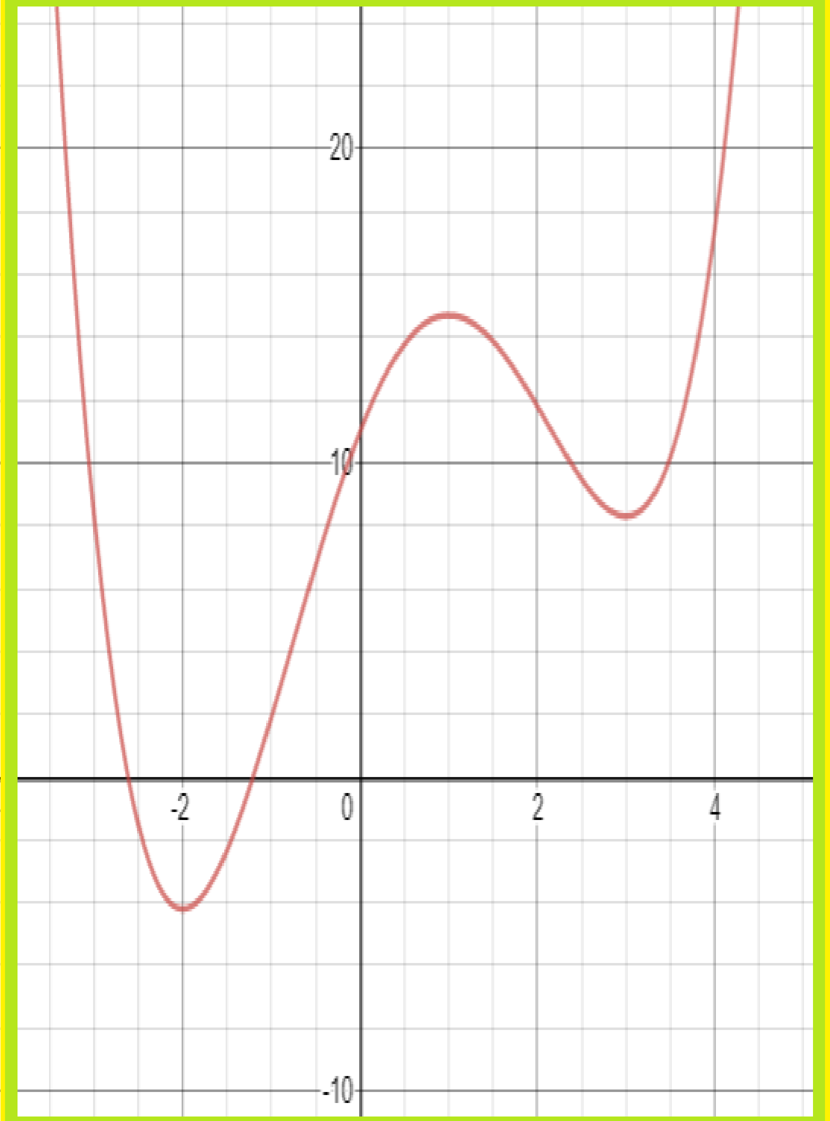
Thus, we get four disjoint intervals

i.e. $(-\infty, -2)$, $(-2, 1)$,$(1 , 3)$, $(3 , \infty)$

Interval	Sign of $f'(x) = \frac{6}{5}(x-1)(x+2)(x-3)$	Nature of $f(x)$
$x \in (-\infty, -2)$	$(-)(-)(-) = (-)$	Strictly decreasing
$x \in (-2, 1)$	$(-)(+)(-) = (+)$	Strictly Increasing
$x \in (1, 3)$	$(+)(+)(-) = (-)$	Strictly decreasing
$x \in (3, \infty)$	$(+)(+)(+) = (+)$	Strictly increasing

$\Rightarrow$ $f(x)$ is strictly decreasing on the interval $x \in (-\infty, -2) \cup (1, 3)$

$f(x)$ is strictly increasing on the interval $x \in (-2, 1) \cup (3, \infty)$



HOME WORK - MIS EX: 3,4,5,6,7